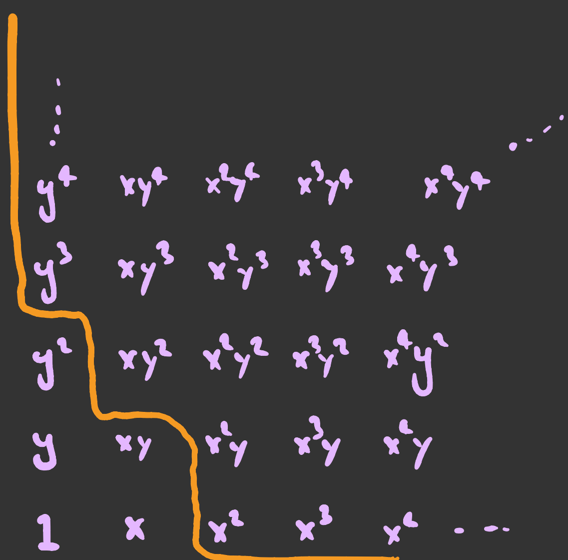
Drawing Rings

Consider the k-algebra $A = k[x,y]$.

We can draw this example very nicely, it becomes a useful testing ground for theorems.

We write out a k-basis



$I = \langle x^2, xy, y^3 \rangle$

def: We define a monomial ideal to be generated by monomials.

We can draw these ideals easily.

Then $\dim A/I$ = # monomials under orange line.

$= 5$

For the next definitions, A any ring.

<u>def^{n}</u>: An A-module F is FREE if

$\exists$ a basis of F, that is $\{x_i \in F\}_{i\in I} =: \mathcal{B}$ s.t.

$\forall x \in F \ \exists a_1, \dots, a_n$ & $x_{i_1}, \dots, x_{i_n} \in \mathcal{B}$ with

$$x = a_1 x_{i_1} + \cdots + a_n x_{i_n}$$

& if for any $x_{i_1}, \dots, x_{i_n} \in \mathcal{B}$ s.t.

$$a_1 x_{i_1} + \cdots + a_n x_{i_n} = 0 \Rightarrow a_1 = \cdots = a_n = 0 .$$

<u>Remark</u>: Once we've defined DIRECT sum,

we see F free $\Leftrightarrow$ $F \cong \bigoplus_{i\in I} A_i$.

$$=: A^{\oplus I}$$

We say a module M is FINITELY GENERATED

if $\exists \ A^{\oplus n} \twoheadrightarrow M$.

↳ finite direct sum.

Now an ideal, that is, a submodule $I \subseteq A$, is finitely generated $\Leftrightarrow$ $\exists f_1, \dots, f_r \in A$ s.t. $I = (f_1, \dots, f_r)$.

"Hilbert's Basis Thm"

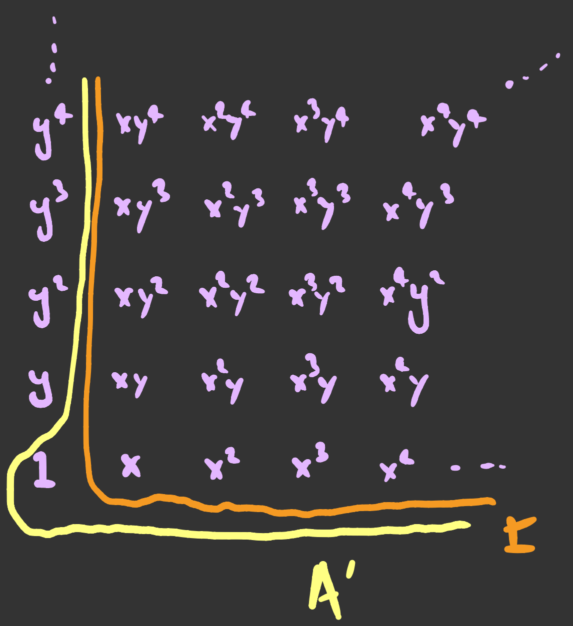
> **IMPORTANT FACT:** Every ideal of $k[x_1, \dots, x_n]$ is finitely generated, we'll prove this later.

def^{n}: A k-algebra A is finitely generated if $\exists a_1, \dots, a_n \in A$ s.t. $\forall a \in A$ $\exists p(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ s.t. $a = p(a_1, \dots, a_n)$.

Equivalently, $\exists$ $k[x_1, \dots, x_n] \twoheadrightarrow A$, a surj. k-algebra homomorphism.

THERE'S A BIG DIFFERENCE BETWEEN FIN. GEN. AS MODULE VS FIN. GEN. AS A k-ALGEBRA.

Now our example:



$I = \langle x \rangle$, is fin. generated

$A' \subseteq A$ is a k-subalgebra.

Then A is fin. generated.

A' is not fin. generated.

Now consider group $G = \{1, -1\} \curvearrowright A = k[x,y]$

via $(-1) \cdot f(x,y) = f(-x,-y)$.

We want to understand

$$A^G = \{ f \in A \mid g \cdot f = f \quad \forall g \in G \}.$$

$$\begin{array}{cccccc}
\vdots & & & & & \iddots \\
y^4 & xy^4 & x^2y^4 & x^3y^4 & x^4y^4 & \\
y^3 & xy^3 & x^2y^3 & x^3y^3 & x^4y^3 & \\
y^2 & xy^2 & x^2y^2 & x^3y^2 & x^4y^2 & \\
y & xy & x^2y & x^3y & x^4y & \\
1 & x & x^2 & x^3 & x^4 & \cdots
\end{array}$$

These monomials form a basis for A^G.

$$A^G = k[x^2, xy, y^2]$$

$$\cong \frac{k[u,v,w]}{(uw - v^2)}$$

So we see A^G is fin. gen.

Moreover, A^G is a free module over $k[x^2, y^2]$ with basis $1, xy$.

$$\begin{array}{cccccc}
\vdots & & & & & \iddots \\
y^4 & xy^4 & x^2y^4 & x^3y^4 & x^4y^4 & \\
y^3 & xy^3 & x^2y^3 & x^3y^3 & x^4y^3 & \\
y^2 & xy^2 & x^2y^2 & x^3y^2 & x^4y^2 & \\
y & xy & x^2y & x^3y & x^4y & \\
1 & x & x^2 & x^3 & x^4 & \cdots
\end{array}$$