ICMS LECTURE 1

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1. Lecture 1: Lattice Polytopes and Cones

The aim of this lecture is to cover the following topics:

(1) Basic definitions and examples
(2) Properties of Polytopes
(3) Newton polytopes and related geometry

Our main references will be the first chapters of [BR15] and Chapter 2.2 in [CLS11].

1.1. **Basics.**

Definition 1.1. A subset $P \subset \mathbb{R}^n$ is a *rational polytope* if

$$P = \mathrm{Conv}(S) := \{\lambda_1 m_1 + \cdots + \lambda_r m_r \mid \lambda_i \geq 0,\ \sum_i \lambda_i = 1\},$$

where $S = \{m_1, \ldots, m_d\} \subset \mathbb{Q}^n$ is a finite set.

A subset $C \subset \mathbb{R}^n$ is called a *rational cone* if

$$C = \mathrm{Cone}(S) = \{\lambda_1 m_1 + \cdots + \lambda_r m_r \mid \lambda_i \geq 0\},$$

where S, as before, is a finite set. If additionally $S \subset \mathbb{Z}^n$ then we replace *rational* with *lattice*.

This is the so called "V-description" of a polytope, the V standing for vertices, which we will formally define late in Definition 1.5. There is another description as a bounded intersection of half-spaces, the so-called "H-description" which we will shortly see.

Remark 1.2. Due to time restrictions, we will only talk about polytopes in this lecture, however the following basic definitions (for example, faces, dimension, and many more) can be made for cones. We refer to [CLS11, Chapter 1] for a thorough treatment.

Definition 1.3. Given codimension 1 hyperplanes H_i in $\mathbb{R}^n$ defined by normal vectors $u_i \in \mathbb{Q}^n$, then the intersection of all their positive half-spaces H_i^+ is a *rational polyhedron.*

We specify the normal vector, so that the *positive* half-space is well defined.

From now we assume that everything is rational and stop writing it. Note that a polyhedron can be unbounded. We then have the following important theorem.

Theorem 1.4. *A bounded polyhedron is a polytope and any polytope is a bounded polyhedron.*

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Let's unpack this. Given a polyhedron $P = \cap_i H_i^+ \subset \mathbb{R}^n$ which is bounded, there exists a finite set $S \subset \mathbb{Q}^n$ such that $P = \mathrm{Conv}(S)$. Conversely, given a polytope $P = \mathrm{Conv}(S)$, then there exist half-spaces H_i^+ such that P is the intersection of all H_i^+.

There is no easy way to write down algorithm to go from one to the other, nor is there an easy proof. We refer to [BR15] for a nice run through of this. However, there do exist algorithms already implemented in `sage` and `polymake` which we will use as we frequently bounce between the two descriptions.

A hyperplane is completely described by a normal vector $u \in \mathbb{Q}^n$ and a number $b \in \mathbb{Q}$:

$$H_{u,a} = \{m \in \mathbb{R}^n \mid \langle m, u \rangle = b\}$$

and the associated half-space is

$$H_{u,a}^+ = \{m \in \mathbb{R}^n \mid \langle m, u \rangle \geq b\}.$$

We define a *supporting hyperplane* of a polytope P to be a codimension 1 hyperplane $H \subset \mathbb{R}^n$ such that

$$P \subset H^+.$$

With this we can define faces of the polytope.

Definition 1.5. Given a polytope P, a *face* of P is a subset $\sigma \subset P$ which is the intersection of P with a supporting hyperplane, that is:

$$\sigma = P \cap H.$$

Note that the face of a polytope is again a polytope.

Definition 1.6. The *dimension* of a polytope P is the dimension of the smallest affine subspace $\mathbb{R}^m \subset \mathbb{R}^n$ containing P.

While developing the theory we will usually assume that $P \subset \mathbb{R}^n$ is *full dimensional*, that is $\dim P = n$. There's normally no harm in this, as when P is not full dimensional, we can simply move P so that it contains the origin, so that the smallest affine subspace containing P is a vector subspace and consider P to be full dimensional in this subspace. There are many cases however, when we have to pay attention to whether or not the polytope is full dimensional or not. Luckily, this property is easily checkable.

Definition 1.7. Consider a polytope P. A k-dimensional face of P is called a *k-face.* A 0-face is called a *vertex.* A $(\dim P - 1)$-face is called a *facet.*

Denote the number of k-faces by $f_k(P)$. We collect this data in the *f-vector.*

$$f(P) = (f_0(P), \ldots, f_{\dim P}(P))$$

Remark 1.8. When P is full dimensional, we can use the facets to give a unique H-description of our polytope. Indeed, each facet $F \subset P$ has a unique supporting hyperplane $H_F = H_{u_F, a_F}$ where $F = H_F \cap P$ and (u_F, a_F) is unique up to multiple by a positive number.

Given a polytope P, its lattice points $P \cap \mathbb{Z}^n$ are of fundamental importance.

Definition 1.9. Given a polytope $P = \mathrm{Conv}(m_1, \dots, m_r)$ and an integer $t \in \mathbb{Z}_{>0}$, we define the t^{th} dilation of P to be $tP := \mathrm{Conv}(tm_1, \dots, tm_r)$. Using this definition we define the lattice point counting function

$$L_p(t) := |tP \cap \mathbb{Z}^n|,$$

that is the number of lattice points in the t^{th} dilation of P.

Proposition 1.10. *Given a lattice polytope $P \subset \mathbb{R}^n$, then $L_P(t) \in \mathbb{Q}[t]$ that is, the lattice point counting function is a polynomial with rational coefficients. Moreover, its degree as a polynomial is* $\dim P$*.*

In light of this last proposition, we call $L_P(t)$ the *Ehrhart polynomial* after Eugène Ehrhart, who proved the main structure theorem concerning these functions [BR15]. The study of these objects can be tracked back to Euler, although he probably did not think of himself as studying lattice point counting functions.

Definition 1.11. We define the *Ehrhart Series* to be

$$\mathrm{Ehr}_P(z) = \sum_{t \geq 0} L_P(t) z^t.$$

Using the Ehrhart series of a polynomial we can define another very important invariant of a polytope. In order to do this, we need to express the Ehrhart series as a fraction of rational polynomials. This is a basic technique from the theory of *generating series* which is an extremely rich area. See [BR15, Chapter 1].

Definition 1.12. h^*-vector - to be done.

1.2. **Examples.** Let's have a look at some basic examples.

Definition 1.13. Let $n \geq 1$, we define the standard n-simplex $\Delta_n := \mathrm{Conv}(e_0, \dots, e_n) \subset \mathbb{R}^{n+1}$.

Fix $n = 1$ and write $\Delta = \Delta_1$ for the 1-simplex, which is just the line segment connecting $(1, 0)$ and $(0, 1)$. Let's compute the invariants above.

In this case, it's really quite easy to see from the pictures that $L_\Delta(t) = t + 1$ and thus $\mathrm{Ehr}_\Delta(z) = \sum_{t \geq 0}(t + 1)z^t$. However, let us do this using a technique which will useful going forward.

You will show in Exercise 1 that $\frac{1}{1-z} = 1 + z + z^2 + \cdots$ in the ring of formal series. Thus, in $\mathbb{Z}[[z]]$, we have that

$$\begin{aligned} \frac{1}{(1-z)^2} &= (1 + z + z^2 + \cdots) \cdot (1 + z + z^2 + \cdots) \\ &= 1 + 2z + 3z^2 + \cdots + (t + 1)z^t + \cdots \\ &= \mathrm{Ehr}_\Delta(z) \ . \end{aligned}$$

This is no coincidence! The coefficient for z^t is exactly the number of ways we can make up t from two smaller numbers (keeping order in mind), that is, $t = 0 + t = 1 + (t - 1) = 2 + (t - 2) = \cdots = t + 0$, corresponding to lattice points $(0, t), (1, t - 1), \dots, (t, 0)$.

In the exercises we will compute $L_{\Delta_n}(t)$ and $\mathrm{Ehr}_{\Delta_n}(z)$ using this method for a general $n > 0$.

1.3. **Properties.** Now we will discuss a few more important properties of polytopes.

Definition 1.14. A polytope $P \subset \mathbb{R}^n$ has the *Integer Decomposition Property*, the IDP for short, if for every $k \in \mathbb{N}$ and $p \in kP \cap \mathbb{Z}^n$ there exist $x_1, \ldots, x_k \in P \cap \mathbb{Z}^n$ such that $p = x_1 + \cdots + x_k$.

This means that every lattice point of every dilation decomposes as a sum of lattice points of P. A word of warning, this property has a few different names in the literature, in particular, a *normal* polytope is a polytope that possesses the IDP.

This property is very important. It tells you something about the geometry of the associated toric variety (which we will explore later) and has important implications for integer programming (which we won't explore later).

This fundamental property remains largely mysterious and there are many open questions as to whether or not classes of well known polytopes have the IDP. There are a few positive results, one of which is the following.

Theorem 1.15. *For every $k \geq \dim P - 1$, the dilation kP has the IDP.*

Note that in general if kP has the IDP, that is not necessarily the case for $(k+1)P$, see [CHHH14].

Corollary 1.16. *Every polytope of dimension 2 has the IDP. (In particular, there is no toric embedding of a toric surface which is not projectively normal.)*

Definition 1.17. We say a polytope $P \subset \mathbb{R}^n$ is *very ample* if there exists a $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ and every $p \in kP \cap \mathbb{Z}^n$, there exist $x_1, \ldots, x_n \in P \cap \mathbb{Z}^n$ such that $p = x_1 + \cdots + x_k$.

So we see immediately that (IDP) $\implies$ very ample and that, contrary to the IDP, if kP is very ample, then $(k+1)P$ is also very ample.

Note that the converse does not hold, that is, there exists polytopes which are very ample but don't have the IDP [LMe17]. Also, it follows immediately from Theorem 1.15, that for each $k \geq \dim P - 1$, we have kP is very ample.

Now we give a few equivalent definitions of the above properties. First, we give a useful and visual criterion for the IDP by considering the cone over the polytope

$$C(P) = \mathrm{Cone}((\{1\} \times P) \cap (\mathbb{Z} \times \mathbb{Z}^n)) \subset \mathbb{R} \times \mathbb{R}^n \quad .$$

Lemma 1.18. *A polytope has the IDP if and only if for every $(k, p) \in C(P) \cap \mathbb{Z}^{n+1}$ there exists $(1, x_1), \ldots, (1, x_k) \in \{1\} \times P \cap \mathbb{Z}^{n+1}$ such that $(k, p) = \sum(1, x_i)$.*

The proof is left as an easy exercise.

Now we give an alternative formulation of very ample. We first need a quick definition.

Definition 1.19. To each vertex $v \in P$, we define the *vertex semigroup*

$$S_v := \mathbb{N}((P \cap \mathbb{Z}^n) - v) \subset \mathbb{Z}^n.$$

Then we say that S_v is *saturated* if whenever $kp \in S_v$ for some $k \in \mathbb{N}$ and $p \in \mathbb{Z}^n$, then $p \in S_v$.

Proposition 1.20. *A polytope P is very ample if and only if the semigroup generated at each vertex is saturated. Equivalently, the Hilbert basis for $C(P)$ is contained inside $\{1\} \times P \cap \mathbb{Z}^{n+1}$.*

We can then further define the *holes* of a given polytope

$$h(P) = C(P) \setminus \langle \{1\} \times P \cap \mathbb{Z}^{n+1} \rangle.$$

The holes keep track of how far our polytope is from having the IDP. Higashitani has some results on the number of holes [Hig14].

Remark 1.21. Note that all the definitions so far are invariant under translation. So that, especially for the exercises, you can translate your polytope to a chosen place in the lattice, often this means moving the origin inside the polytope (If that's possible).

Now we define the polar dual and use it to define *reflexivity*, our next property of interest.

Definition 1.22. Let P be a lattice polytope. Define the *polar dual* of P to be

$$P^* = \mathrm{Conv}(\frac{1}{a_F} u_F \mid F \text{ a facet})$$

Note that $P^{**} = P$.

A polytope is called *reflexive* if P^* is a lattice polytope.

Proposition 1.23. *A polytope is reflexive if and only if the origin is the unique interior lattice point.*

1.4. **Newton Polytopes and Toric Geometry.** We now shift our perspective. We think of $\mathbb{N}^n \subset \mathbb{Z}^n$ as the space of all exponent vectors, that is, we identity a point $\underline{m} \in \mathbb{N}^n$ with its corresponding monomial $x^{\underline{m}} = x_1^{m_1}, \ldots, x_n^{m_n}$.

Definition 1.24. Given a polynomial $f \in k[x_1, \ldots, x_n]$ we write it as

$$\sum_{\underline{m} \in \mathbb{N}^n} c_{\underline{m}}(f) x^{\underline{m}},$$

where only finitely many of the coefficients are non-zero. We define its *support* to be $\mathrm{Supp}(f) = \{\underline{m} \in \mathbb{N}^n \mid c_{\underline{m}}(f) \neq 0\}$.

The support of a polynomial are the monomials which appear in f. Using our new perspective, we consider this as a set of lattice points.

Definition 1.25. Given a polynomial $f \in k[x_1, \ldots, x_n]$, we define its *Newton Polytope* to be

$$\mathrm{NP}(f) = \mathrm{Conv}(\mathrm{Supp}(f)).$$

We say that f is saturated in its Newton Polytope if $\mathrm{NP}(f) \cap \mathbb{Z}^n = \mathrm{Supp}(f)$, we often abbreviate this to SNP.

Given a very ample polytope $P \subset \mathbb{Z}^n$, we can define its associated toric variety as follows. Let us denote the lattice points of P by $A = P \cap \mathbb{Z}^n$ and consider vector space $\mathbb{C}^A$ indexed by A. Then we can define

$$\begin{aligned} \Phi_A : &(\mathbb{C}^*)^n \longrightarrow \mathbb{P}(\mathbb{C}^A) \\ &(a) \longmapsto [(a^{\underline{m}})_{\underline{m} \in A}] \ . \end{aligned}$$

Then we can define

$$X_A := \overline{\Phi_A((\mathbb{C}^*)^n)} \subset \mathbb{P}(\mathbb{C}^A) \ .$$

This is the toric variety associated to P.

Example 1.26. Take $P = d\Delta_1 \subset \mathbb{R}^2$. Then

$$\Phi_A : (x_0, x_1) \longmapsto (x_0^d : x_0^{d-1} x_1 : \cdots : x_1^d)$$

is the Veronese embedding of $\mathbb{P}^1$. This generalises to give that $d\Delta_n$ gives the d^{th} Veronese embedding of $\mathbb{P}^n$.

The best way to understand this is through loads of examples. We will continue with this later.

Remark 1.27. Note that it is not obvious at all why we required P to be very ample. Indeed, the above construction works for any polytope (actually any point configuration A) and we get a toric variety. However, the theory is much better behaved when we take A to be the lattice points of a very ample polytope. That's all we will say at this point, see [CLS11] for the canonical reference and [GKZ08, Chapter 5] for a very good but brief introduction.

Potential future topics

The theorem of Hochster? The following is highly non-trivial.

Theorem 1.28. *Every toric variety is Cohen-Macaulay.*

Constructions of non-very ample and non-normal simplices?

Post lecture exercise

In this exercise you will compute the Ehrhart polynomial and series of two polytopes. Confirm your answers using a math software system of your choice (for example, `sage` or `polymake`).

Let $P = \texttt{cube}(n) = \prod_1^n [0,1]$.

(1) Prove that $L_P(t) = (t+1)^n$. Write out the expansion in terms of binomial coefficients.

(2) Prove that

$$(-1)^n L_P(-t) = L_{P^\circ}(t) \ .$$

(3) Show that the Ehrhart Series can be described as follows:

$$\mathrm{Ehr}_P(z) = \frac{1}{z} \sum_{t \geq 0} t^n z^t \ .$$

(4) Let $f(P) = (f_0, \dots, f_n)$ be the f-vector. Using a computer, compute $f_k \cdot 2^{k-n}$ for a few numbers. Prove a formula for f_k in terms of n and k.

Exercises to Lecture 1

(1) Let $P = \Delta_n = \mathrm{Conv}(e_0, \dots, e_n) \subset \mathbb{R}^{n+1}$. Compare with Example 1.2 in the lectures for the case $n = 2$.

(a) Show that $(1 - z)(1 + z + z^2 + \cdots + z^n) = 1 - z^{n+1}$.

(b) Show that in the power series ring $k[[z]]$ that we have

$$\frac{1}{1-z} = \sum_{i \geq 0} z^i \ .$$

(c) Using the computation in the last exercise, write out the expansion of $\frac{1}{(1-z)^n} \in k[[z]]$ and conclude that

$$\mathrm{Ehr}_P(z) = \frac{1}{(1-z)^n} \ .$$

Now compute the Taylor expansion around 0 (assume $k = \mathbb{C}$ if you like) and show that

$$L_P(t) = \mathrm{const} \binom{n+t}{t} \ .$$

(2) Let $f, g \in \mathbb{C}[x_1, \dots, x_n]$. Prove that

$$\mathrm{NP}(f \cdot g) = \mathrm{NP}(f) + \mathrm{NP}(g) \ ,$$

where the right hand side denotes the *Minkowski sum* of two polytopes.

(3) Prove that a one-dimensional Lattice polytope has the IDP.

(4*) Prove that a lattice polytope of dimension d such that each edge has at least d lattice points is very ample.

Hint: Use the saturated semi-group definition of very ampleness. Center each vertex at the origin, and try and fit a small polytope into that vertex, such that several (how many?) dilations are still inside your big polytope.

References

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