SS 25 | TU Berlin
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ALGEBRA 2
ÜBUNGSBLATT 1

(1) (a) Describe the zero-divisors, nilpotent elements and the units in the ring $\mathbb{Z}/36\mathbb{Z}$.
(b) Show that for a finite ring A, that is $|A| < \infty$, each element is either a unit or a zero divisor.
(c) Show by an example that this fails to be true for infinite rings.

Definition: An ideal $\mathfrak{p} \subset A$ is called *prime* if $A/\mathfrak{p}$ is an integral domain. An ideal $\mathfrak{m} \subset A$ is called *maximal* if $A/\mathfrak{m}$ is a field. Compare with [Atiyah-MacDonald, page 3].

(2) (a) Let $k[X]$ denote the polynomial ring with variable X over a field k. Show that every nonzero prime ideal of $k[X]$ is a maximal ideal.
(b) Is this still true if we replace k by $\mathbb{Z}$?

(3) Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

(4) Prove that the following are equivalent:
(a) A has exactly one prime ideal.
(b) every element is either a unit or nilpotent.
(c) $A/\mathfrak{N}$ is a field. (Recall $\mathfrak{N}$ is the nilpotent radical.)

(5) Let k be an infinite field and consider a polynomial $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$. Prove that $f(a_1, \dots, a_n) = 0$ for every $(a_1, \dots, a_n) \in k^n$ if and only if $f = 0$ is the zero polynomial.

Hint: Note that for one variable, this is just the statement that a univariate polynomial has only finitely many roots. Then consider f as a polynomial in $n-1$ variables over the ring $k[x_n]$, and argue via induction.