ALGEBRA 2

ÜBUNGSBLATT 10

(1) Let $X = \mathrm{Spec}(A)$ and $Y \subset X$ be a subset. Show the following.
   (a) Let $x = \mathfrak{p}_x \in X$ be a point. Then $\overline{\{x\}} = V(\mathfrak{p}_x)$. Conclude that x is a *closed point* if and only if $\mathfrak{p}_x$ is maximal.
   (b) $V(I(Y)) = \overline{Y}$.
   (c) Y is dense in X if and only if $I(Y) = \mathcal{N}(A)$, the nilpotent radical of A.
   (d) Let X be irreducible. Every non-empty open subset U of X is dense.
   (e) Given a multiplicatively closed set $S \subset A$. Then $S^{-1}\,\mathrm{rad}(\mathfrak{a}) = \mathrm{rad}(S^{-1}\mathfrak{a})$.

(2) Let B be a flat A-algebra via the morphism $\varphi : A \to B$. Show that the following are equivalent.
   (a) $\mathfrak{a}^{ec} = \mathfrak{a}$ for all ideals $\mathfrak{a}$.
   (b) $\varphi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.
   (c) For every maximal ideal $\mathfrak{m}$ it holds that $\mathfrak{m}^e \neq (1)$.
   (d) If M is any non-zero module, then $M_B \neq 0$.
   (e) For every A-module M, the mapping

$$M \longrightarrow M_B, \quad x \longmapsto 1 \otimes x$$

   is injective.
   Hint: See the hint accompanying Exercise 3.16 in [AM].

(3) Let $f : B \to B'$ be a morphism of A-algebras and let C be an A-algebra. Show that if f is integral, then $f \otimes \mathrm{id}_C : B \otimes_A C \to B' \otimes_A C$ is integral.

(4) Let $A \subseteq B$ be rings with B integral over A. Show that
   (a) If $x \in A$ is a unit in B, then it is a unit in A.
   (b) The Jacobson radical of A is the contraction of the Jacobson radical of B.

(5) Prove that the ring extension $\mathbb{Z} \subset \mathbb{Z}[\frac{1}{2}]$ is not integral. Prove that for $\alpha = \sqrt{2} + \sqrt{3}$, the ring extension $\mathbb{Z} \subset \mathbb{Z}[\alpha]$ is integral.
   Hint: It suffices to show α is integral over $\mathbb{Z}$.