ALGEBRA II
ÜBUNGSBLATT 11

(1) Let k be a field and $A = k[x_0, \ldots, x_n]$ be the polynomial ring. Take some prime $\mathfrak{p} \in \operatorname{Spec} A$ and $g \notin \mathfrak{p}$.

Prove that $\frac{f}{g} \in A_{\mathfrak{p}}$ defines a function

$$\frac{f}{g} : U \to k$$

where $U \subset k^n$ is a Zariski open set. Show that this open set intersects $V(\mathfrak{p}) \subset K^n$ in a dense open subset of $V(\mathfrak{p})$.

(2) Draw a picture of the algebraic set $V_{\mathbb{R}}(y^2 - x^3 - x^2) \subseteq \mathbb{R}^2$.

(3) Let k be an algebraically closed field.
  (a) Find $f \in k[T]$ such that $I(V((T-1)^2)) = (f)$.
  (b) Take $f_1, \ldots, f_s \in k[T_1, \ldots, T_n]$ irreducible polynomials and $e_1, \ldots, e_s \in \mathbb{Z}_{>0}$. Prove that
$$I(V(f_1^{e_1} \cdots f_s^{e_s})) = (f_1 \cdots f_s).$$

(4) Let k be an algebraically closed field and consider a polynomial $f \in A = k[x_0, \ldots, x_n]$ such that $\deg f > 0$. We call an algebraic set of the form $V(f) \subset k^n$ a *hypersurface*.
  (a) Given some other $g \in A$ such that $f \nmid g$, show that $V(f) \not\subset V(g)$.
  (b) Deduce that irreducible components of a hypersurface $V(g)$ are of the form $V(f_i)$ where $g = u \cdot \prod f_i$ for some unit $u \in k^{\times}$.

(5) (a) Let Y be an algebraic set. Prove that $Y \subseteq k^n$ is irreducible if and only if $I(Y)$ is prime.
  (b) Let $X = k$, where k is any field. Prove that $Y \subseteq X$ is closed if and only if Y is finite. Is every non-empty open subset dense?

(6) Let $f : A \to B$ be a ring homomorphism. Prove that the induced map on spectra

$$f^* : \operatorname{Spec} B \to \operatorname{Spec} A, \quad \mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$$

is a closed map. That is, sends closed sets to closed sets.