SS 25 | TU Berlin
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ALGEBRA II
ÜBUNGSBLATT 12

(1) Let $A = \mathbb{Z}[x]$.

(a) Show that

$$\operatorname{Spec} A = \{(f) < \mathbb{Z}[x] \mid f \text{ irreducible } \} \cup \operatorname{Spmax} \mathbb{Z}[x],$$

where $\operatorname{Spmax} A$ is the set of maximal ideals.

(b) prove that each $\mathfrak{m} \in \operatorname{Spmax} A$ has the form $\mathfrak{m} = p, g$ where $p \in \mathbb{Z}$ is a prime number and $g \in A$ is a polynomial whose reduction modulo p is an irreducible element $\overline{g} \in \overline{\mathbb{F}}_p[x]$.

(c) Conclude the quotient $A/\mathfrak{m}$ is a finite algebraic field extension of $\mathbb{F}_p$

(d) Deduce a similar result for $A = k[x, y]$, where k is not necessarily an algebraically closed field.

(2) Let M be an A-module and $u : M \to M$ an endomorphism. Show that:

(a) If u is surjective and M noetherian, then u is an isomorphism.

(b) If u is injective and M artinian, then u is an isomorphism.

(c) If M has finite length, then there is a natural number n such that $M = \ker(u^n) \oplus \operatorname{im}(u^n)$.

Definition: A module M is called indecomposable if $M \neq 0$ and given any decomposition $M = M_1 \oplus M_2$, then either $M_1 = 0$ or $M_2 = 0$.

(d) Assume that M is indecomposable and of finite length. Prove that u is either nilpotent or an automorphism.

(3) Let M be an A-module and N_1, N_2 submodules of M. Show that if M/N_1 and M/N_2 are noetherian (respectively artinian), then $M/(N_1 \cap N_2)$ is noetherian (respectively artinian).

(4) Let M be a noetherian A-module and $\mathfrak{a} = \operatorname{ann}(M)$ its annihilator. Show that the ring $A/\mathfrak{a}$ is also noetherian.

(5) Let $A = \prod_{n \geq 0} \mathbb{Z}/2\mathbb{Z}$ be the ring with componentwise addition and multiplication.

(a) Prove that ideals of A are of the form $\prod_{n \geq 0} \mathfrak{a}_n$, where $\mathfrak{a}_n$ is an ideal of $\mathbb{Z}/2\mathbb{Z}$. Characterise the prime ideals of A.

(b) Show A is not noetherian, but the set of *prime* ideals satisfies the ascending chain condition.

(c) For each prime $\mathfrak{p}$ in A the local ring $A_{\mathfrak{p}}$ is noetherian. (Hint: use the fact that $x(x-1) = 0$ for all $x \in A$.)