SS 25 TU Berlin
Prof. Peter Bürgisser Institut für Mathematik
Dr. Dominic Bunnett 7. Mai 2025

ALGEBRA 2
ÜBUNGSBLATT 3

(1) The ring of Gaussian integers $B = \mathbb{Z}[i]$ is defined as the subring of $\mathbb{C} = \mathbb{R}[i]$ given by the elements $x = a + bi$ with $a, b \in \mathbb{Z}$. Show that B together with the norm function $N(x) = a^2 + b^2$ is a Euclidean ring.

That is, for all $x, y \in B$ with $y \neq 0$ there are $q, r \in B$ with $x = qy + r$ and $N(r) < N(y)$. In particular, B is a principal ideal domain and a UFD. Describe all the units in B.

(2) (a) How are prime ideals of the product ring $A \times B$ related to the prime ideals of A and B?
(b) Let $n > 1$ be an integer. Determine all prime ideals of the residue class ring $\mathbb{Z}/(n)$.

(3) Let $\mathfrak{a} \leq A$ be an ideal and write $\mathfrak{a}[x] \leq A[x]$ for the extension of $\mathfrak{a}$ under the inclusion homomorphism $A \hookrightarrow A[x]$.
(a) Prove that $\mathfrak{p}[x]$ is prime if $\mathfrak{p}$ is prime.
(b) Is $\mathfrak{m}[x]$ maximal if $\mathfrak{m}$ is maximal?

The prime spectrum of a ring
Recall that $\mathrm{Spec}(A) = \{\mathfrak{p} \leq A \mid \mathfrak{p}\,\text{prime}\}$ and that for a subset $E \subseteq A$ we defined

$$V(E) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid E \subseteq \mathfrak{p}\}.$$

Convince yourself that the sets $V(E)$ satisfy the axioms for the closed sets of a topology. The resulting topology is the *Zariski Topology.*

(4) Draw pictures of $\mathrm{Spec}\,\mathbb{Z}$, $\mathrm{Spec}\,\mathbb{C}$ and $\mathrm{Spec}\,\mathbb{C}[x]$.

(5) For each $f \in A$, denote $X = \mathrm{Spec}\,A$ and $X_f = \mathrm{Spec}(A) - V(f)$. Show that
(a) $X_f \cap X_g = X_{fg}$;
(b) $X_f = \emptyset \iff f$ is nilpotent;
(c) $X_f = X \iff f$ is a unit;
(d) $X_f = X_g \iff \mathbf{r}((f)) = \mathbf{r}((g))$;
(e) X is quasi-compact (that is, every open covering has a finite subcovering).
(f)* Show that the sets X_f form a basis for the Zariski topology.