SS 25
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# ALGEBRA 2

## ÜBUNGSBLATT 5

**Remark.** Let

$$\begin{array}{ccccccc} & & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & & \end{array}$$

be a commutative, exact diagram. Then as in the snake lemma [AM, Prop. 2.10], with precisely the same proof, there is an exact sequence

$$\ker(f') \to \ker(f) \to \ker(f'') \xrightarrow{d} \operatorname{coker}(f') \to \operatorname{coker}(f) \to \operatorname{coker}(f'').$$

(1) Let $P$ be an $A$-module. Show that the following three conditions involving $A$-modules and $A$-linear maps are equivalent.

- (a) For any $g : M \to M''$ surjective and $h : P \to M''$, there is $\tilde{h} : P \to M$ such that $g\tilde{h} = h$.
- (b) Any short exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} P \to 0$ splits.
- (c) There is an $N$ such that $P \oplus N$ is free.

Modules $P$ satisfying these conditions are called projective.

(2) Let $A$ be a local ring with maximal ideal $\mathfrak{m}$ and residue class $k = A/\mathfrak{m}$. Let $M$ be a finitely generated $A$-module. Prove that if $M$ is projective then it is free.

*Hint:* Let $x_1, \ldots, x_n \in M$ be such that the classes $\bar{x}_1, \ldots, \bar{x}_n \in M/\mathfrak{m}M$ form a $k$-basis. Then $x_1, \ldots, x_n$ is a generating system of $M$, and it is minimal in the sense that any proper subset does not generate.

(3) Let $f : M \to N$ be a morphism of $A$-modules.

- (a) Show that the kernel of $f$ is an $A$-module $K$ together with a morphism $i : K \to M$ which is up to (unique) isomorphism uniquely determined by the following "universal property":
  - (i) $f \circ i = 0$.
  - (ii) For any morphism $j : L \to M$ with $f \circ j = 0$ there is a unique morphism $h : L \to K$ such that $j = i \circ h$.
- (b) Show that any morphism $i : K \to M$ satisfying (a) and (b) is injective.
- (c) Formulate a similar property for the cokernel. Compare this with the induced morphism property [AM, p. 19] (Homomorphiesatz auf Deutsch).

(4) Let $A$ be a non-zero ring. Consider an $A$-linear map $\phi : A^n \to A^m$.

(a) Show that if $\phi$ is surjective then $m \leq n$.

(b) Show that if $\phi$ is an isomorphism then $m = n$.

(c) If $\phi$ is injective, is it always the case that $n \leq m$?

*Hint:* Let $\mathfrak{m}$ be a maximal ideal and $\phi : A^n \to A^m$ be an isomorphism. Consider tensoring the morphism $\phi$ by $A/\mathfrak{m}$.

(5) Let $M$ be an $A$-module. $M$ is called *finitely presented* if there is an exact sequence $F_1 \to F_0 \to M \to 0$ with $F_0, F_1$ finitely generated free $A$-modules (hence $F_0 \cong A^{n_0}, F_1 \cong A^{n_1}$ for $n_0, n_1 \in \mathbb{N}$).

Prove that if $M$ is a finitely presented and if there is a short exact sequence $0 \to K \to N \to M \to 0$ with $N$ finitely generated, then $K$ is finitely generated as well.

*Hint:* Construct a commutative diagram of the following form:

$$\begin{array}{ccccccccc} & & A^{n_1} & \longrightarrow & A^{n_0} & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_1} & & \downarrow & & \| & & \\ 0 & \longrightarrow & K & \xrightarrow{f} & N & \xrightarrow{g} & M & \longrightarrow & 0. \end{array}$$