ALGEBRA 2
ÜBUNGSBLATT 7

(1) *(Vanishing Criterion for tensors)* Consider two A-modules M and N and $\{n_i\}_{i\in I}$ a generating set of N. Show that for every $t \in M \otimes N$ there exist $m_i \in M$ for each $i \in I$ with only finitely many non-zero such that

$$t = \sum m_i \otimes n_i.$$

Conclude that $t = 0$ if and only if there exists $a_{ij} \in A$ for some indexes $j \in J$ such that $m_i = \sum_J a_{ij} m_j$ and $\sum_I a_{ij} n_i = 0$.

(2) Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module, form the B module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.
Hint: Consider the morphism $p : N_B \to N$ defined by sending $b \otimes y = by$.

(3) Let M be an A-module. Prove that if M is flat, then the scalar extension $B \otimes_A M$ is a flat B-module.

Hint: Show that a generalised associativity of the tensor product holds: Suppose that M is an A-module, N a B-module and P an $A-B$-bimodule (that is, P is simultaneously an A-module and a B-module, and these structures are compatible in the sense that $a(xb) = (ax)b$ for all $a \in A, b \in B$ and $x \in P$ holds). In a canonical way, $M \otimes_A P$ is a B-module and $P \otimes_B N$ an A-module and there is an isomorphism

$$(M \otimes_A P) \otimes_B N \cong M \otimes_A (P \otimes_B N).$$

The proof follows a very similar shape to that of the standard associativity of the tensor product [AM, Prop. 2.14ii].

(4) Let A be an integral domain and M an A-module. An element $x \in M$ is a *torsion element* if $\mathrm{Ann}(x) \neq 0$, that is, if x is killed by some non-zero element of A. Show that the torsion elements of M form a submodule. This submodule is called the torsion submodule of M and denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

(a) If M is any A-module, then $M/T(M)$ is torsion-free.
(b) If $f : M \to N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.
(c) if $0 \to M' \to M \to M''$ is exact, then $0 \to T(M') \to T(M) \to T(M'')$ is exact.

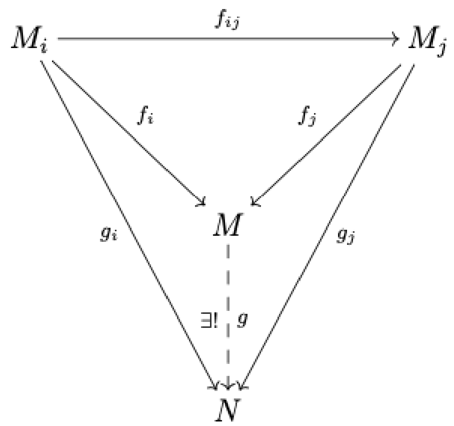
(d) If M is any A-module, then $T(M)$ is the kernel of the mapping $M \to K \otimes M, \quad x \mapsto 1 \otimes x$, where K is the field of fractions of A.
Hint: Use that K may be regarded as the direct limit of its submodules $A\xi$ for $\xi \in K$.

Direct Limits. Let $(I, \leq)$ be a partially ordered set which is *directed*, that is, for all $i, j \in I$ there exists some $k \in I$ such that $i, j \leq k$. For each $i \in I$ let M_i be an A-module and for all $i \leq j$ let $f_{ij} : M_i \to M_j$ be a morphism such that the following is satisfied:

- $f_{ii} = \mathrm{id}_{M_i}$ for all $i \in I$.
- for all $i \leq j \leq k$ we have $f_{ik} = f_{jk} \circ f_{ij}$.

We refer to (M_i, f_{ij}) as a direct system of A-modules over I. A module M together with morphisms $f_i : M_i \to M$ for each i is called a direct limit of this direct system and denoted by $\varinjlim M_i$ if the following universal property is satisfied:

- $f_i = f_j \circ f_{ij}$ for all $i \leq j$.
- For any A-module N and morphisms $g_i : M_i \to N$ with $g_i = g_j \circ f_{ij}$ there is a unique morphism $g : M \to N$ such that $g \circ f_i = g_i$ for all $i \in I$.



Then M is (up to isomorphism) uniquely determined by this property and the existence is shown by the following construction: Denote by $\iota : M_i \to \bigoplus_{l \in I} M_l$ the canonical embedding.

Define $M = \left(\bigoplus_{i \in I} M_i\right) / U$ where U is the submodule U generated by all elements of the form $\iota_i(x) - \iota_j \circ f_{ij}(x)$ for all $x \in M_i$ and $i \leq j$. For each i let $f_i : M_i \to M$ be $f_i = p \circ \iota_i$, with p the canonical surjection modulo U.

(5) (a) Show that the above construction of M and the f_i yields indeed a direct limit of the direct system (M_i, f_{ij}).

(b) Show that every element $x \in \varinjlim M_i$ can be written as $x = f_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.

(c) Show that if $f_i(x_i) = 0$, then there is $j \leq i$ such that $f_{ij}(x) = 0$.

(d) Let $\{M_i \subset M\}_{i \in I}$ be a system of submodules of a fixed module M and assume that for all i, j there is a k such that $M_i + M_j \subset M_k$. We define a partial order on I by declaring that $i \leq j$ if $M_i \subseteq M_j$ and denote the inclusions by f_{ij}. Show that $\varinjlim M_i$ agrees with $\sum_{i \in I} M_i$. Conclude that every A-module is a direct limit of finitely generated modules.

(e) Show that each module is a direct limit of finitely presented modules.

(f*) In which way is the direct sum of modules a slight variation of a direct limit? (It is not precisely a direct limit in the sense defined above).