SS 25 | TU Berlin
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ALGEBRA 2
ÜBUNGSBLATT 8

(1) Let M be a finitely generated A-module. Let S be a multiplicative set. Show that $S^{-1}M = 0$ if and only if there exists an $s \in S$ such that $sM = 0$.

(2) Let S be a multiplicative set in a ring A and let M, N be A-modules. Consider the map

$$S^{-1}\operatorname{Hom}_A(M,N) \to \operatorname{Hom}_{S^{-1}A}\left(S^{-1}M, S^{-1}N\right)$$

defined by $f \mapsto S^{-1}f$ (see [AM, page 38, last paragraph]). Show that

(a) this map is injective if M is finitely generated.

(b) Moreover, it is bijective if M is finitely presented.

Hint: Use [AM, Prop. 3.3, Prop. 3.5, Prop. 2.9].

(3) Let A be a ring and M a finitely presented A-module. Prove:

(a) M is projective if and only if $M_{\mathfrak{m}}$ is projective (= free) over $A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ in A.

Hint: To prove the 'reverse' direction, use Exercise 2. More precisely, use the fact that for every maximal ideal $\mathfrak{m}$ it holds that

$$\operatorname{Hom}_A(M,N)_{\mathfrak{m}} \cong \operatorname{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}).$$

Then prove that projectivity can be expressed as follows:

An A-module P is projective if and only if for every surjective morphism between two A-modules $\varphi : M \to N$ the induced morphism

$$\varphi^* : \operatorname{Hom}_A(P,M) \to \operatorname{Hom}_A(P,N), \quad h \mapsto \varphi \circ h,$$

is surjective (equivalently the functor $\operatorname{Hom}_A(P,-)$ is right exact).

(b) M is projective if and only if M is flat.

(4) Let S be a multiplicatively closed subset of an integral domain A. Show that $S^{-1}T(M) = T(S^{-1}M)$ for any A-module M. Prove that being torsion-free is a local property, that is, that the following statements are equivalent:

(a) M is torsion-free.

(b) $M_{\mathfrak{p}}$ is torsion-free for every prime ideal $\mathfrak{p}$.

(c) $M_{\mathfrak{m}}$ is torsion-free for every maximal ideal $\mathfrak{m}$.

Coproducts

Recall the following definition from the exercise session. Given a family of objects in a category $\{X_i\}_{i\in I}$ we define the *coproduct* of this family in $\mathcal{C}$ to be the following data and universal product:

(i) A object $X = \coprod_{i\in I} X_i$ and maps $\iota_i : X_i \to X$.
(ii) Given an object Y and a family of morphisms $\{f_i : X_i \to Y\}$ there is a unique morphism $f : X \to Y$ such that $f_i = f \circ \iota_i$.

We proved in the lecture that if a coproduct exists, then it is unique up to unique isomorphism.

(5) Given two A-algebras B and C. Prove that

$$B \coprod C = B \otimes_A C.$$

That is, prove that the coproduct of B and C is the tensor product in the category of A-algebras.

$$\begin{tikzcd} X_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & X \arrow[d, "\exists! f"] \\ & Y \end{tikzcd}$$