ALGEBRA 2
ÜBUNGSBLATT 9

(1) Let V be a finite dimensional k-vector space, where k is a field. Pick some element $\varphi \in \mathrm{End}_k(V)$ and consider the k-algebra morphism

$$k[x] \longrightarrow \mathrm{End}_k(v), \quad x \longmapsto \varphi.$$

Denote the kernel $I_\varphi = (p_\varphi)$ and consider the ring

$$A_\varphi = k[x]/I_\varphi.$$

(a) What is p_φ known as in Linear Algebra?
(b) Take $\overline{k} = k$ to be algebraically closed. Exhibit a bijection between $\mathrm{Spec}(\varphi)$, consisting of all eigenvalues and $\mathrm{Spec}(A_\varphi)$.
(c) To which 'parts' of p_φ does $\mathrm{Spec}(A_\varphi)$ correspond if k is not necessarily algebraically closed?

(2) (a) Given a ring homomorphism $\varphi : A \to B$, show that the map

$$\varphi^* : Y = \mathrm{Spec}(B) \longrightarrow X = \mathrm{Spec}(A), \quad \mathfrak{p} \longmapsto \mathfrak{p}^c = \varphi^{-1}(\mathfrak{p})$$

is well-defined and that if $f \in A$, then $(\varphi^*)^{-1}(X_f) = Y_{\varphi(f)}$.

Remark: This implies that φ^* is a continuous map.

(b) Consider a ring A and a multiplicatively closed subset S of A. Let $\phi : A \to S^{-1}A$ be the canonical ring homomorphism. Show that $\phi^* : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is a homeomorphism onto its image in $X = \mathrm{Spec}(A)$. In particular, prove that if $f \in A$, then $\mathrm{Spec}(A_f)$ in X is the basic open set X_f (compare with the third exercise sheet).
(c) Conclude that any $X_f \subseteq X$ with the subspace topology is quasi-compact.

Definition A topological space X is *irreducible* if for every decomposition $X = X_1 \cup X_2$ with X_1, X_2 closed subsets of X, then $X = X_1$ or $X = X_2$.

For example, the space $X = \mathrm{Spec}(k[x,y]/(xy))$ is *not* irreducible as

$$X = V(x) \cup V(y) = \mathrm{Spec}(k[x,y]/(x)) \cup \mathrm{Spec}(k[x,y]/(y)).$$

(3) Let $f : X \to Y$ be a continuous map between topological spaces. Show that for every irreducible subspace $V \subset X$ the image $f(V)$ in Y is irreducible.

(4) Let $X = \mathrm{Spec}(A)$. Show that

(a) $Y \subset X$ is irreducible if and only if $I(Y)$ is a prime ideal of A, where $I(Y) = \cap_{\mathfrak{p} \in Y} \mathfrak{p}$.

(b) Conclude that X is irreducible if and only if the nilradical $\mathcal{N}(A)$ is a prime ideal if and only if there is a unique minimal prime ideal $\mathfrak{p}$.

Suppose that $\mathfrak{p}$ is indeed the unique minimal prime. Must we have $\mathfrak{p} = 0$?

(5) Let $X = \mathrm{Spec}(A)$ and $Y \subset X$ be a subset. Recall $V(E) := \{\mathfrak{p} \in \mathrm{Spec}\, A \mid E \subset \mathfrak{p}\}$. Show the following:

(a) Let $x = \mathfrak{p}_x \in X$ be a point. Then $\overline{\{x\}} = V(\mathfrak{p}_{\mathfrak{x}})$. Hence x is a *closed point* if and only if $\mathfrak{p}_x$ is maximal.

(b) $V(I(Y)) = \overline{Y}$.

(c) Y is dense in X if and only if $I(Y) = \mathcal{N}(A)$, the nilpotent radical of A.

(d) Let X be irreducible. Every non-empty open subset U of X is dense.