ALGEBRA 2
ÜBUNGSBLATT 4

(1) Consider $\varphi : \mathbb{Z} \to \mathbb{Z}[i]$. We showed last week, using the norm function, that $\mathbb{Z}[i]$ is a UFD.

You may use without proof that for a prime number p, we have that

$$p \equiv 1 \mod 4 \iff p = a^2 + b^2, \quad \text{for some } a, b \in \mathbb{Z}.$$

Prove the following:

(a) The norm function is multiplicative.
(b) The extension of a prime ideal along φ may not be prime.
(c) If $p \equiv 1 \mod 4$ then $(p)^e$ is the product of two distinct prime ideals.
(d) If $p \equiv 3 \mod 4$ then $(p)^e$ is prime in $\mathbb{Z}[i]$.

(2) Consider a ring A such that for every $x \in A$ it holds that $x^2 = x$. Prove

(a) $2x = 0$ for every $x \in A$;
(b) every prime ideal $\mathfrak{p}$ is maximal and $A/\mathfrak{p}$ is a field with two elements;
(c) every finitely generated ideal is principal.

Suppose now that A is such that for every $x \in A$ we have $x^n = x$. Prove that every prime ideal $\mathfrak{p}$ is maximal. Prove that $A/\mathfrak{p}$ is a finite field and find a bound for the number of elements.

(3) Consider an exact sequence of A-modules

$$0 \to M' \to M \to M'' \to 0.$$

Show that if M' and M'' are finitely generated, then so is M. Does the converse hold?

(4) Let A be a ring and $\mathfrak{a}$ be an ideal contained in the Jacobson radical of A. Moreover, let M be an A-module, N be a finitely generated A-module, and $u : M \to N$ be A-linear. Show that if the induced map $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.

(5) Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of A-modules. Show that the following two conditions are equivalent.

(a) There is a morphism $s : M'' \to M$ such that $gs = \mathrm{id}_{M''}$.

(b) There is a morphism $p : M \to M'$ such that $pf = \mathrm{id}_{M'}$.

If there conditions hold, we say that the sequence splits. Show that in this situation, we have

$$M = \mathrm{im}(s) \oplus \ker(g) = \mathrm{im}(f) \oplus \ker(p) \cong M' \oplus M''.$$