ALGEBRA 2
ÜBUNGSBLATT 6

(1) Let M be an A-module. Show that:
   (a) If M is projective, then it is flat.
   (b) The converse holds if A is local and M is finitely presented.
   Hint: Show directly that M is free.

   Remark (b) is also true for non-local rings, which will follow from localisation. This will appear in a future exercise sheet.

(2) Let $(M_i)_{i\in I}$ be a family of A-modules, Show:

$$\bigoplus_{i\in I} M_i \text{ is flat } \iff M_i \text{ is flat for all } i \in I.$$

(3) (Universal property of scalar extension.) Let $f : A \to B$ be a ring homomorphism. Let M be an A-module and $\varphi : M \xrightarrow{1\otimes \mathrm{id}_M} M_B$. Show that for every B-module N and any A-linear map $\psi : M \to N$ there is a unique B-linear map $\gamma : M_B \to N$ such that $\psi = \gamma \circ \varphi$.

   In other words, prove that

$$\mathrm{Hom}_B(B \otimes_A M, N) \xrightarrow{\cong} \mathrm{Hom}_A(M, N), \quad h \mapsto h \circ \varphi$$

   is a (natural[1]) isomorphism.

(4) Let N be an A-module and consider a short exact sequence of A-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

   Prove that after apply Hom one gets the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(N, M') \longrightarrow \mathrm{Hom}_A(N, M) \longrightarrow \mathrm{Hom}_A(N, M'').$$

   Show, by providing an example, that the last map in the above sequence is not necessarily surjective.

[1] Ignore this word if it doesn't make sense to you.

(5) Let M_1, M_2 and N_1, N_2 be A-modules.

(a) Prove that there is an A-linear map

$$\mathrm{Hom}_A(M, M') \otimes_A \mathrm{Hom}_A(N, N') \to \mathrm{Hom}_A(M \otimes_A N, M' \otimes_A N')$$

which sends a simple tensor $\varphi \otimes \psi$ to the linear map $\varphi \otimes \psi$.

(b) Prove that this is an isomorphism if all modules are finitely generated and free.

(c) Let $A = k$ be a field and take M and N finite dimensional. For $\varphi \in \mathrm{Hom}_k(M, M)$ and $\psi \in \mathrm{Hom}_k(N, N)$ show that

$$\mathrm{tr}(\varphi \otimes \psi) = \mathrm{tr}(\varphi)\,\mathrm{tr}(\psi) \quad \text{and} \quad \det(\varphi \otimes \psi) = \det(\varphi)^{\dim N} \det(\psi)^{\dim M}.$$

Hint: Show and use that $\varphi \otimes \psi = (\varphi \otimes \mathrm{id}_N) \circ (\mathrm{id}_M \otimes \psi)$.

Definition: Let A be a ring and B an A-algebra. Define the module of Kähler differentials of B over A as follows

$$\Omega^1_{B/A} := \left(\bigoplus_{f \in B} B \cdot df \right) \Big/ R \; .$$

This is a quotient of a free B-module (generated by symbols df for each element $f \in B$) by the submodule R, generated by the following relations:

(a) $d(f+g) = df + dg$ (additivity)

(b) $d(fg) = f(dg) + g(df)$ (product rule)

(c) $da = 0$ for every $a \in A$ (zero on ‘constants’).

(6*) (a) Consider $A = k$ and $B = k[x, y]$. Show that $d(3xy^2) = 3y^2 \cdot dx + 6xy \cdot dy$. In general, show that for any $f \in B$, we have that $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$.

Hint: Prove this for monomials via the product rule and induction, then use additivity.

Prove that

$$\Omega^1_{k[x,y]/k} \cong B \cdot dx \oplus B \cdot dy.$$

(b) Consider a field extension L/K and consider L as a K-algebra. Prove that L/K is separable if and only if $\Omega^1_{L/K} = 0$.